Contaminant diffusion in a random array of fixed parallel cylinders at high Reynolds numbers

Piero Olla
The James Franck Institute and Ryerson Laboratories, The University of Chicago, Chicago, Illinois 60637
(Received 18 February 1994)

A model for the high Reynolds number flow in a fixed bed of parallel cylinders of radius d and mean separation L is described, and predictions for the diffusion coefficients of a contaminant, passively advected by the flow, are provided in the dilute limit $d/L \to 0$. The longitudinal diffusion appears to be dominated by the wakes and to be much larger than the transversal part, which is dominated by turbulence. The basic result is that increasing the spacing between cylinders results in an increase rather than in a decrease of the amount of diffusion, with scaling $(dL^2)^{1/3}$ both in the longitudinal and in the transversal directions. The diffusion becomes normal at length scales of the order of L^2/d.

PACS number(s): 47.27.Qb, 05.40.+j

I. INTRODUCTION

There are several circumstances in natural environments in which one has to deal with high Reynolds numbers flows in very complicated or even random geometries. A river flowing through a brush of reeds and the wind blowing in a forest are two typical examples, in which the complicated geometry is provided by a random distribution of cylindrical obstacles. An important problem, with practical application to the study of contaminants in natural environments, is the dynamics of passive scalars. Of course, during the years, a huge amount of work has been done on turbulent diffusion in such complex environments (see, for instance, [1] for a review). The basic difficulty is that one has to deal with a situation of inhomogeneous turbulence in which there are very complicated interactions between the mean flow and the turbulent fluctuations. As regards the particular issue of contaminant dynamics, this problem is made particularly interesting by the coexistence and interaction of two different mechanisms for dispersion, such as turbulence and the random environment. This paper is devoted to the study of this interaction and to provide some estimates for the diffusion constant in turbulent flows through fixed, dilute beds of infinitely long, parallel cylinders.

Flows in fixed beds have long been considered, in the limit of zero Reynolds numbers, as a model for porous media. What one is dealing with in this case is a Stokes flow, which is fully determined in terms of the no-slip boundary conditions on the surface of the obstacles. The simplest possible description of this problem is that of Brinkman [2], in which the effect of the obstacles is considered in an average sense as an additional drag term in the Stokes equation. In the work of Hinch [3], a formal derivation of this approximation was obtained and later, Koch and Brady [4] used the same formalism to derive models of diffusion in porous media. More recently, Kaneda [5] used the formalism of Hinch to calculate the correction to the Brinkman equation produced by finite inertia. This amounted to taking into account the presence of laminar wakes in which the flow is nonpotential.

The transition to the high Reynolds number limit is characterized by the wakes becoming turbulent. The presence of turbulence makes the exact treatment of the high Reynolds limit impossible. At the same time, the complicated geometry makes an asymptotic analysis of the kind "homogeneous-isotropic" turbulence impossible as well. However, it is hoped that detailed information on the structure of turbulence is not needed and that what is really necessary are local estimates of the fluctuation amplitude, and of the typical eddy size. Unfortunately, this is not completely true, since some information on the energy spectrum is needed to estimate the eddy viscosity at small scales, which is necessary in turn to calculate the velocity defect at various positions in the wakes. It will turn out that the present model is able, even within a simple mixing length approximation, to obtain an estimate of the turbulent energy spectrum.

To go back to a simpler case of the Stokes flow, the idea of Hinch was basically to calculate the modification to the mean flow produced by a given obstacle, keeping to lowest order, and in an average sense, the coupling with the perturbation produced individually by each other obstacle. This was essentially a mean field theory approach, in which the averages were carried on over an ensemble of exact bed configurations. Keeping this mean field effect produced the drag contribution to the flow equation of the Brinkman approximation. In the large Reynolds number limit, however, this is likely not to be the only effect of a mean field analysis.

In a high Reynolds number regime, a first average over turbulent fluctuations is necessary to separate out the mean flow for a given exact bed configuration. Once this is done, the mean flow will depend on an equation in which the Newtonian viscosity is negligible compared to the turbulent strain. Here, this turbulent strain will be parametrized in terms of an eddy viscosity. When deriving the equation for the wake produced by a given obstacle, the eddy viscosity will depend on the turbulence level in the wake under examination and on the turbulence from the other wakes. Hence, besides producing a drag term, at high Reynolds numbers, the mean field effect of the obstacles is a contribution to the eddy viscosity, which has to be taken into account when calculating the wake profiles.

What is going to be derived here is a Brinkman approximation for high Reynolds number flows in fixed beds of infinitely long cylinders. This is a limit that is difficult to find realized in nature, but which is interesting because it allows one to study the interaction among different wakes in a purely two dimensional setting. Likewise, any effect coming from the finite extension of the bed perpendicularly to the cylinders will be disregarded. The philosophy that will be followed is that of Hinch [3], with the addition of a number of approximations and assumptions, which are the following.

(1) High Reynolds numbers: the Reynolds number $\mathrm{Re}\sim U_0 d/\nu_0$, with U_0 the mean flow velocity, d the cylinder radius, and ν_0 the Newtonian viscosity is taken large enough (i.e., $\mathrm{Re}>2000$) so as to allow the wakes to become fully turbulent and standard results about the self-similar turbulent wake regime [6] to be at least approximately applicable.

(2) Turbulence model: it is assumed that the mean velocity gradients determine the turbulence levels and that the characteristic length scales of these gradients determine the size of the largest eddies. So, a velocity variation ΔU across a distance l is assumed to generate a turbulent velocity v_T of the same order of magnitude and on the same scale of ΔU. This is essentially a mixing length approximation. The effect of turbulence on the mean flow is parametrized in terms of a scale dependent eddy viscosity, defined in terms of the turbulent velocity difference at scale l: v_l through the equation $\nu_l \sim [\int_0^l dl' l'^2 dv_{l'}^2/dl']^{1/2}$. This becomes asymptotic for $l\to\infty$ to the standard definition of eddy viscosity: $\nu_T \sim v_T l_{\mathrm{eddy}}$, with l_{eddy} the characteristic size of the energy containing eddies.

(3) The drag: Each cylinder is taken to produce a drag per unit length given by $\Delta\sigma \sim U^2 d$, which is equivalent to saying that any fluid element coming in front of a cylinder is virtually stopped by it. Summing over unit area, this produces a drag coefficient in the Navier-Stokes equation: $\xi \sim U_0 d/L^2$, where L is the typical distance between cylinders.

(4) Mean field theory: this is the basic working assumption of the model. As in Hinch [3], this requires that the bed be dilute, with separation distances large compared with the cylinder radii. What it means is that correlations between velocity disturbances produced by different cylinders can be treated in a perturbative fashion. If $v_{1,2}$ are the velocity disturbances produced by cylinders 1 and 2, to lowest order $\langle F[v_1]G[v_2]\rangle = \langle F[v_1]\rangle\langle G[v_2]\rangle$, where F and G are generic functionals and $\langle\ \rangle$ is the combined average over turbulent fluctuations and bed configuration. This approximation is meaningful when the wake produced by a given cylinder is affected by the effect of the wakes from many cylinders upstream, so that the global effect can be considered in an average sense [7].

This paper is organized as follows. In the next section the equations governing the diffusion of a passive scalar in the random bed are introduced and all basic quantities defined. It will appear that an important role is played by the mean velocity profile in a wake (averaged over both bed configurations and time). In Sec. III, the average wake profile and turbulence levels are derived using the mean field approximation. Estimates for the diffusion coefficients (including the skewness) are presented in Sec. IV. Section V contains a discussion of the results.

II. TRACER DYNAMICS

Consider a volume of water flowing with an average velocity $\mathbf{U}_0$ perpendicular to a fixed bed of infinitely long, parallel cylinders, all with the same radius d. Let us take a coordinate system with the x axis directed along the mean flow and the cylinders along the z axis. The cylinders are distributed with a density $n(\mathbf{r}) = \sum_i \delta(\mathbf{r}-\mathbf{r}_i)$. Here of course, $\mathbf{r}_i$ is the coordinate of the ith cylinder. Following Hinch [3], we can introduce a hierarchy of partial averages; let us indicate with $\langle\ \rangle_{ij\ldots}$ the average over turbulence and over the position of all cylinders but the ones at $\mathbf{r}_i,\mathbf{r}_j,\ldots$, that are kept fixed. In this way, we can introduce the coarse grained density $\bar{n}$: $\bar{n}=\langle n\rangle = L^{-2}$, with L typical distance between cylinders, and other statistical quantities, such as, for instance, the conditional probability $P(\mathbf{r}_1|\mathbf{r})$ to find a cylinder in $\mathbf{r}$, given the presence of another one in $\mathbf{r}_1$, which is defined through the expression $\langle n(\mathbf{r})\rangle_1 = \bar{n}P(\mathbf{r}_1|\mathbf{r})$.

The fluid velocity can be divided into a time independent part $\mathbf{U}$ and a turbulent component $\mathbf{v}$. Again following Hinch [3], the time independent part can be expanded in terms of contributions from higher correlations:

$$\mathbf{U}(\mathbf{r}) = \mathbf{U}_0(\mathbf{r}) + \mathbf{U}_1(\mathbf{r}) + \mathbf{U}_2(\mathbf{r}) + \cdots , \tag{1a}$$

where

$$\mathbf{U}_1(\mathbf{r}) = \sum_i \mathbf{U}_1(\mathbf{r}_i|\mathbf{r}) , \quad \mathbf{U}_2(\mathbf{r}) = \sum_{i,j} \mathbf{U}_2(\mathbf{r}_i,\mathbf{r}_j|\mathbf{r}) , \tag{1b}$$

and similar expressions for the higher order terms. The term $\mathbf{U}_1(\mathbf{r}_i|\mathbf{r})$ in Eq. (1b) is the disturbance to the velocity field due to a cylinder in $\mathbf{r}_i$, disregarding the effect of all the other cylinders. Similarly $U_2(\mathbf{r}_i,\mathbf{r}_j|\mathbf{r})$ is the disturbance produced to $U_1(\mathbf{r}_i|\mathbf{r})$, by the presence of a cylinder at $\mathbf{r}_j$, but disregarding the effect of all others. Notice that to lowest meaningful order, from Eqs. (1a) and (1b), one has also

$$\begin{aligned} &\langle \mathbf{U}\rangle \simeq \mathbf{U}_0 , \quad \langle \mathbf{U}(\mathbf{r})\rangle_i \simeq \mathbf{U}_0 + \mathbf{U}_1(\mathbf{r}_i|\mathbf{r}) , \\ &\langle \mathbf{U}(\mathbf{r})\rangle_{ij} \simeq \mathbf{U}_0 + \mathbf{U}_1(\mathbf{r}_i|\mathbf{r}) + \mathbf{U}_1(\mathbf{r}_j|\mathbf{r}) + \mathbf{U}_2(\mathbf{r}_i,\mathbf{r}_j|\mathbf{r}) . \end{aligned} \tag{2}$$

The spatial inhomogeneity of the corrections to the mean velocity, $\mathbf{U}_i(\mathbf{r})$, $i=1,2,\ldots$, is a source of turbulence in the problem, so that it is possible to expand $\mathbf{v}$ in a way similar to that of (1a):

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_1(\mathbf{r},t) + \mathbf{v}_2(\mathbf{r},t) + \cdots . \tag{3}$$

In principle it would be possible to carry on a decomposition of v in terms of contributions from the individual wakes, in the line of Eq. (1b), but this appears to be useless, except near the cylinders, where the turbulence from the wake dominates over that from the background. Definitions similar to those given for the velocity are adopted for the pressure P. The equations describing the behavior of the various quantities appearing in Eqs. (1)–(3) will be derived in the next section.

To lowest order, the diffusion of a tracer particle with respect to the mean flow is calculated starting from the equation

$$\mathbf{r}_1(t)\simeq\int_0^t dt_1\left\{\mathbf{v}_1(\mathbf{r}(t_1),t_1)+\int d^2r'n(\mathbf{r}')\mathbf{U}_1(\mathbf{r}'|\mathbf{r}(t_1))\right\}, \tag{4}$$

where $\mathbf{r}(t)=\mathbf{U}_0t+\mathbf{r}_1(t)$. Complete information on the evolution of the concentration of tracer particles $\rho(\mathbf{r},t)$ could be obtained imposing conservation of probability: $\rho(\mathbf{r}(\mathbf{r}_0,t),t)=\rho(\mathbf{r}_0,0)$, where $\mathbf{r}(\mathbf{r}_0,t)$ is the coordinate of the particle that at time $t=0$ was at $\mathbf{r}_0$ (Lagrangian coordinate). An alternative would be to follow Koch and Brady [4] and to write the fluid equations for the transport of the tracer, with a diffusion term accounting for the effect of turbulence and of the wakes. This is the second moment of the tracer distribution starting from a point source at an initial time. However, approximate, analytical expression can be obtained for generic moments of the displacement $\mathbf{r}_1$, which could prove useful, due to the non-Gaussian nature of the concentration profiles seen in experiments [8]. If the contaminant density varies little on scales smaller than the cylinder separation, it is possible to equate the average over tracer particles with the average over cylinder position with respect to a given tracer particle. From Eq. (4), one obtains then the expression for the mean drift:

$$\langle\mathbf{r}_1(t)\rangle\simeq\int_0^t dt_1\int d^2r'\langle n(\mathbf{r}')\mathbf{U}_1(\mathbf{r}'|\mathbf{r}(t_1))\rangle\ , \tag{5}$$

while for the next moment, one has

$$\langle\mathbf{r}_1(t)\mathbf{r}_1(t)\rangle\simeq\int_0^t dt_1\int_0^t dt_1\left\{\langle\mathbf{v}_1(\mathbf{r}(t_1),t_1)\mathbf{v}_1(\mathbf{r}(t_2),t_2)\rangle+\int d^2r'\int d^2r''\langle n(\mathbf{r}')n(\mathbf{r}'')\mathbf{U}_1(\mathbf{r}'|\mathbf{r}(t_1))\mathbf{U}_1(\mathbf{r}''|\mathbf{r}(t_2))\rangle\right\}\ . \tag{6}$$

The analysis can be simplified if the integrals in Eqs. (5) and (6) are dominated by contributions at large distance from the cylinders, in the range in which $U_0\gg U_1\gg U_{1_y}$. In this case, the diffusion from the wakes [the piece proportional to $\mathbf{U}_1\mathbf{U}_1$ in Eq. (6)] is in the direction of the unperturbed flow $\mathbf{U}_0$. Using the following expression for the two-point density correlation:

$$\langle n(\mathbf{r})n(\mathbf{r}')\rangle=\bar{n}\delta(\mathbf{r}-\mathbf{r}')+\bar{n}^2P(\mathbf{r}|\mathbf{r}')\ , \tag{7}$$

the part of the diffusion due to the wakes is given by

$$\langle\Delta x(t)^2\rangle_w\simeq\int_0^t dt_1\int_0^t dt_2\int d^2r'\int d^2r''\{\bar{n}\delta(\mathbf{r}'-\mathbf{r}'')+\bar{n}^2[P(\mathbf{r}'|\mathbf{r}'')-1]\}$$
$$\times\int d^2r_1\int d^2r_2 g(\mathbf{r}',\mathbf{r}_1,t_1)g(\mathbf{r}'',\mathbf{r}_2,t_2)U_1(\mathbf{r}'|\mathbf{r}_1)U_1(\mathbf{r}''|\mathbf{r}_2)\ , \tag{8}$$

where $g(\mathbf{r}',\mathbf{r},t)$ is the Green function for the tracer particle motion. For $x-x'\gg d$, i.e., away from the cylinders, one has $x(t)\simeq U_0t$, while the transversal displacement $y(t)\simeq y_1(t)$ is mainly due to turbulent diffusion [the contribution from the transverse drift U_{1_y} is killed by a factor $d/(x-x')$]. This leads to a Green function in the form

$$g(\mathbf{r}',\mathbf{r},t)=\frac{\delta(x-U_0t)}{\sqrt{2\pi D_lt}}e^{-y^2/2D_lt}\ ,$$

where D_l is the turbulent diffusion at scale l, with $l=l(x-x')$ the width at $\mathbf{r}$ of a wake generated by a cylinder at $\mathbf{r}'$. If the cylinders are distributed at random, $P(\mathbf{r}'|\mathbf{r}'')\neq 1$, only when $|\mathbf{r}'-\mathbf{r}''|\ll L$, and the contribution proportional to $\bar{n}^2$ can be neglected in Eq. (8). An order of magnitude estimate of $\langle\Delta x(t)^2\rangle$ can be obtained by neglecting the transverse diffusion in the expression for the Green function g, which is equivalent to taking $\mathbf{r}(t)=\mathbf{U}_0t$ in Eqs. (5) and (6), so that $y_1=y_2$. (In the analysis of Koch and Brady [4], this would correspond to calculating the one-cylinder disturbance to density and fluid velocity, using unperturbed trajectories.) The physical basis of this approximation is that a fluid particle captured by the turbulent wake at a given time tends to remain inside it afterwards (the turbulent wake profile is essentially the envelope of the most typical trajectories of the fluid elements entrained by the wake). We obtain therefore

$$\langle\Delta x(t)^2\rangle_w\simeq 2\bar{n}\int_0^{x(t)}dx_1\int_0^{x_1}dx_2\int_{-\infty}^{x_2}dx'\int_{-\infty}^{+\infty}dy'U_0^{-2}U_1(\mathbf{r}'|\mathbf{r}_1)U_1(\mathbf{r}'|\mathbf{r}_2)\ . \tag{9}$$

Similar calculations are carried on in the Appendix to derive an expression for the skewness. Now, if the approximation $x(t)\simeq U_0 t$ were relaxed, among the other things, the term U_0^{-2} in (9) should be substituted by one in the form

$$[(U_0+U_1+v)(x'|x_1)(U_0+U_1+v)(x'|x_2)]^{-1} .$$

Since right behind the cylinders, $U_0+U_1\rightarrow 0$ and there is almost no turbulence, it appears that there is a possibly important contribution to the transport from the very near wakes, not taken into account by (9), which has the form, first of a sink of contaminant, and then of a source, as the contaminant is first trapped by the regions immediately behind the cylinders and then released. A discussion of this effect will be presented in Sec. IV.

In order for Eq. (9) to make sense at all, the integrals to the right hand must be dominated by the far wake region: $x_{1,2}-x'\gg d$. By conservation of momentum, the velocity gap $U_s(x-x')=U_1(x'|x)$ and the wake width $l(x)$ are connected by the relation $l(x)U_s(x)\simeq$const, where the deviation from equality is produced by the presence of the drag and is negligible up to distances of the order of $l_\xi=U_0/\xi$, with ξ the drag coefficient in the Navier-Stokes equation (typically $l_\xi\gg L$). If $l(x)\sim x^\gamma$, with $\gamma<1$, it appears therefore that Eq. (9) is dominated by the far wake. Power counting suggests that this is the situation also for higher moments. For an isolated cylinder, at sufficient distance from the cylinder $(x>20d)$, dimensional analysis gives the result $l(x)\propto\sqrt{dx}$ [5], which suggests that the condition $\gamma<1$ might be satisfied also in the case of a cylinder bed. If this is true, it is the far wake region that dominates diffusion. Notice that in the absence of drag, the equality $l(x)U_s(x)$=const is exact and for $\alpha<1$ the integrals in Eq. (9) [and (A3) in the Appendix] are divergent at infinity; the presence of the drag, as in the Brinkman approximation [2], will set a cutoff for these integrals at the scale l_ξ.

III. DETERMINATION OF THE WAKE PROFILE

The equations for the mean flow $\mathbf{U}_0$ and the wake profile $U_1(\mathbf{r}'|\mathbf{r})$ are obtained from the Navier-Stokes equation by substituting Eq. (1) and taking, in the first case, a full average $\langle\ \rangle$ and, in the second, a one-cylinder average $\langle\ \rangle_i$. Using Eq. (2) leads then to the equations for the mean flow and the wake profile, which at steady state have the form

$$[\mathbf{U}_0\cdot\nabla+\xi(\mathbf{r})-\nabla\nu(\mathbf{r})\cdot\nabla]\mathbf{U}_0=\mathbf{f}-\nabla P_0(\mathbf{r}) , \qquad (10)$$

with $\mathbf{f}$ an external force to maintain steady state, and

$$\{[\mathbf{U}_0+\mathbf{U}_1(\mathbf{r}_1|\mathbf{r})]\cdot\nabla+\xi(\mathbf{r}_1|\mathbf{r})-\nabla\nu(\mathbf{r}_1|\mathbf{r})\cdot\nabla\}\mathbf{U}_1(\mathbf{r}_1|\mathbf{r}) = -\nabla P_1(\mathbf{r}_1|\mathbf{r}) . \qquad (11)$$

The drag terms ξ and the eddy viscosities ν, appearing in Eqs. (10) and (11), contain the contribution from turbulence and all the higher order correlations. The mean field assumption consists here of using for ξ and ν their lowest order expressions:

$$\xi(\mathbf{r})=\left\langle\int d^2r_1 n(\mathbf{r}_1)\mathbf{U}_0\cdot\nabla\mathbf{U}_1(\mathbf{r}_1|\mathbf{r})\right\rangle , \qquad (12a)$$

$$\xi(\mathbf{r}_1|\mathbf{r})=\left\langle\int d^2r_2 n(\mathbf{r}_2)\mathbf{U}_1(\mathbf{r}_1|\mathbf{r}_2)\cdot\nabla\mathbf{U}_2(\mathbf{r}_1\mathbf{r}_2|\mathbf{r})\right\rangle_1 , \qquad (12b)$$

and

$$\nabla\nu(\mathbf{r})\cdot\nabla\mathbf{U}_0(\mathbf{r})=\langle\mathbf{v}_1\cdot\nabla\mathbf{v}_1\rangle , \qquad (13a)$$

$$\nabla\nu(\mathbf{r}_1|\mathbf{r})\cdot\nabla\mathbf{U}_1(\mathbf{r}_1|\mathbf{r})=\langle\mathbf{v}_1\cdot\nabla\mathbf{v}_1\rangle_1 . \qquad (13b)$$

The only contributions to the integrals for the drag terms in Eqs. (12a) and (12b) come from the cylinder surfaces [3]. In the case of a Stokes flow, the use of no slip boundary conditions allowed the direct calculation of the drag forces. In the present case, this would require us to study the turbulent boundary layer on the surface of the cylinders, which is beyond the capability of the model. Experimental data for the case of an isolated cylinder [6] indicate that the drag per unit length is approximately equal to $U_0^2 d$, with d the cylinder radius. In our case, if the cylinders are not too close, it is reasonable to assume that the boundary layers on the surface of the cylinders are not affected by the turbulence from the other cylinders. Hence, introducing a coefficient $\alpha_\xi\simeq 1$, it is possible to write

$$\xi(\mathbf{r})\simeq\xi(\mathbf{r}_1|\mathbf{r})\simeq\xi\equiv\alpha_\xi U_0 d/L^2 . \qquad (14)$$

The eddy viscosities of Eqs. (13a) and (13b) can be written in the following form:

$$\nu(\mathbf{r})\simeq\alpha_\nu\left[\int_0^l dl' l'^2\frac{d\langle v_{l'}^2\rangle}{dl'}\right]^{1/2} ,$$
$$\nu(\mathbf{r}_1|\mathbf{r})\simeq\alpha_\nu\left[\int_0^l dl' l'^2\frac{d\langle v_{l'}^2\rangle_1}{dl'}\right]^{1/2} , \qquad (15)$$

where the length l in Eqs. (15) refers to the scale of variation of $\mathbf{U}_0$ and $\mathbf{U}_1$, while $\mathbf{v}_l=\mathbf{v}_1(\mathbf{r}+\boldsymbol{l},t)-\mathbf{v}_1(\mathbf{r},t)$ is the turbulent velocity difference at distance l. Taking the limit $l\rightarrow\infty$ in Eq. (15) defines the large scale eddy viscosity: $\nu_T\sim v l_{\mathrm{eddy}}$. Notice that here, variations in the time averaged velocity perturbation $\mathbf{U}_1(\mathbf{r})$ are expected on a smaller scale than l_{eddy}, which is a manifestation of the small scale nature of the forcing. The coefficient α_ν is a constant in the inertial range (in homogeneous isotropic turbulence $\alpha_\nu\simeq 0.3$), but could be dependent on the scale l in general. It will appear, however, that in the $d/L\rightarrow 0$ limit and for large enough Reynolds numbers, the effect of this variation can be neglected.

In our case, $\mathbf{U}_0$ is prescribed, so that Eq. (10) is not necessary. Equation (11) can be simplified, in the limit of large distance from the cylinder, by taking U_1/U_0, $\partial_x U_1/\partial_y U_1$, U_{1_y}/U_1, and $\partial_x P_1$ to be small quantities. A second approximation [6] is to take ν as a constant across a wake, which is equivalent to assuming that wake turbulence is thoroughly mixed at a given distance from the cylinder. Hence $\nu(0|\mathbf{r})\simeq\nu_{l(x)}(x)$, with $l(x)$ the thickness of the wake at distance x from a cylinder. These approximations lead to the equation for $U_s(\mathbf{r})=-U_1(0|\mathbf{r})$:

$$U_0\frac{\partial U_s}{\partial x}-\nu\frac{\partial^2 U_s}{\partial y^2}+\xi U_s=0\ . \tag{16}$$

Using the ansatz $U_s(\mathbf{r})=A(x)\exp[-y^2/2l(x)^2]$, Eq. (16) leads to

$$\frac{1}{2}\frac{dl^2}{dx}=\frac{\nu(l)}{U_0}\ , \tag{17}$$

while integrating over y gives

$$U_s(x)=\alpha_U U_0\Theta(x)\exp(-x/l_\xi)d/l(x)\ , \tag{18}$$

where $l_\xi=U_0/\xi$ and $\alpha_U=\alpha_\xi/\sqrt{2\pi}\simeq 0.4$ for $\alpha_\xi\simeq 1$;$\Theta(x)$ is the Heaviside step function.

In the case of an isolated cylinder, the drag coefficient ξ is zero and the eddy viscosity is due to the turbulence produced by the wake. In this case, a mixing length approximation can be used by setting $v_s=\alpha_v U_s$ with v_s the fluctuating velocity from wake turbulence and α_v and $O(1)$ constant] and setting the typical eddy size to be equal to $l(x)$. Using Eqs. (15) and (18), this leads to a constant value of the eddy viscosity, so that (16) is a heat equation and (17) gives rise to a parabolic wake profile. Experimental data [6] show then that $\alpha_v\simeq 0.6$ and $\nu\simeq 0.13l(x)v_s(x)$, so that

$$l(x)\simeq 0.25\sqrt{dx}\quad\text{and}\quad \nu\simeq U_0 d/32\ . \tag{19}$$

In the case of a bed of cylinders, both coefficients α_ν and α_v are likely to change. In the following analysis it will appear that these constants do not act separately but enter all relevant expressions in the product $\alpha_\nu\alpha_v$.

In the case of an isolated cylinder, the eddy viscosity at a distance x is just $\alpha_\nu l(x)v_s(x)$, with $v_s(x)\sim U_s(x)$. In the presence of other cylinders, the problem is more complicated because, at sufficient distances, the turbulence from the other wakes becomes dominant. Hence it becomes necessary to calculate the fluctuating velocity at scale l from the "background" turbulent field, which requires information about the energy spectrum.

The problem is that, if the range of scales dominated by the background velocity field is long enough, turbulent fluctuations at scale l will receive contribution not only from the spatial gradients of $U_1(\mathbf{r})$ at that scale, but also from the turbulent fluctuations at scales larger than l through the cascade mechanism. Let us introduce the "structure function" for the mean field:

$$U_l^2=\langle U_1(y+l)-U_1(y)]^2\rangle$$
$$=2[\langle\Delta U(0)\Delta U(y)\rangle-\langle\Delta U^2\rangle]\ , \tag{20}$$

where $\Delta U(y)=U_1(y)-\langle U_1\rangle$, and longitudinal variations in the velocity field are neglected. If we assume that in the presence of a single large scale energy source, the turbulent fluctuations would arrange themselves in a Kolmogorov spectrum: $v_l\propto l^{1/3}$, and we take $U_l\propto l^\beta$, there are essentially two possibilities. If $\beta>\frac{1}{3}$, at sufficiently small scales, the cascade will dominate over the source of energy from the mean field at that scale, and the energy spectrum will tend to the Kolmogorov limit. If on the other hand $\beta<\frac{1}{3}$ the opposite situation will occur and $v_l=\alpha_v U_l$.

From Eq. (1b), we obtain for the two-point correlation for δU

$$\langle\Delta U(0)\Delta U(y)\rangle=\alpha_U^2\sqrt{\pi}U_0^2(d/L)^2$$
$$\times\int_0^\infty\frac{dx}{l(x)}\exp[-2x/l_\xi-y^2/4l^2(x)]\ . \tag{21}$$

Cascade dominance can be excluded in the following way. Let $l(x)\propto x^\gamma$ be the leading order behavior for the wake thickness in the range $l_w\ll l\ll l_{\text{eddy}}$, where l_w indicates the thickness of the wake at which the background starts to dominate. Substituting this into $\langle\Delta U(0)\Delta U(l)\rangle$ gives the result, in the limit $l/l_{\text{eddy}}\to 0$,

$$\langle\Delta U(0)\Delta U(l)\rangle\propto\exp[-D(l_\xi,L,d)l^{2/(1+2\gamma)}]\ ,$$
$$\text{so that }\beta=(1+2\gamma)^{-1}\ . \tag{22}$$

Substituting into Eq. (17) and keeping, for $l\gg l_w$, only the contribution from the background, leads then to the equation

$$\frac{dl^2}{dx}\propto\begin{cases}l^{1+\beta}\Longrightarrow\gamma=(1-\beta)^{-1} & \text{if }\beta<\frac{1}{3}\\ l^{4/3}\Longrightarrow\gamma=\frac{3}{2} & \text{if }\beta>\frac{1}{3}\ .\end{cases} \tag{23}$$

Equations (22) and (23) have the only solution: $\gamma=(1+\sqrt{3})/2\simeq 1.37$, which corresponds to $\beta\simeq 0.27<\frac{1}{3}$; we can then set $v_l\simeq\alpha_v U_l$, with α_v hopefully not too strongly dependent on the scale l. The parameter α_v, like α_ν, should, however, tend to a constant in the "inertial range" for the background fluctuations: $l_w<l<l_{\text{eddy}}$, provided a wide enough range of scales is available. The condition $v_l\sim U_l$ has the physical meaning that the component of the background turbulence that affects the most a wake comes from cylinders that are close to the one generating that wake. Including the turbulence of the wake from the fixed cylinder, the expression for v_l must, however, be corrected to the form $v_l^2\simeq v_s(x)^2+\alpha_v^2U_l^2$, where $l=l(x)$ and $v_s(x)$ is the turbulence from the wake. Since in the region in which $v_s(x)$ is dominant the dynamics is essentially that of the wake from an isolated cylinder, I can set $v_s(x)\simeq 0.6U_s(x)$ as in the isolated cylinder case. Substituting into Eq. (15) and using Eq. (18) gives then the result

$$\nu(0|\mathbf{r})\simeq\frac{U_0 d}{32}\left[\exp(-2x/l_\xi)+\frac{\alpha^2}{(U_0d)^2}\int_0^l dl'l'^2\frac{dU_{l'}^2}{dl'}\right]^{1/2}\ , \tag{24}$$

where $\alpha\simeq 32\alpha_v\alpha_\nu\alpha_U$ is equal to one if α_v, α_ν, and α_U have the same value they had in the isolated cylinder case. Substituting into Eq. (17) gives finally the equation for the wake. It is better to introduce new rescaled variables η and F:

$$x = l_\xi \eta/2 \quad \text{and} \quad l^2 = g^{2/3} F L^2/32 , \tag{25}$$

where

$$g \simeq \frac{\alpha^2 \alpha_\xi}{2^{7/2}\pi} \frac{L}{d} . \tag{26}$$

In these new variables, the wake equation reads

$$F'(\eta) = \left[g^{-4/3} e^{-\eta} + \int_0^\infty d\zeta \frac{e^{-\zeta}}{\sqrt{F(\zeta)}} [4F(\zeta)(1 - e^{-F(\eta)/4F(\zeta)}) - F(\eta) e^{-F(\eta)/4F(\zeta)}] \right]^{1/2} . \tag{27}$$

Equation (26) can be solved iteratively and a good approximation can be obtained already at first order in the procedure. For $g = O(1)$, or for small values of η, it appears that the term in $g^{-4/3}$ dominates, so that the expression for F that is appropriate to utilize in the right hand side of (27) is $F(\zeta) = g^{-2/3}\zeta$. For $g \to \infty$, the integral term dominates, so that the right choice is $F(\zeta) = (4\pi)^{1/3}\eta$. The integral in Eq. (27) can then be calculated using the formula $\int_0^\infty dx \exp(-\mu x^2 - a x^{-2}) = \frac{1}{2}\sqrt{\pi/\mu}\exp(-2\sqrt{a\mu})$ [9], leading to the result

$$F(\eta) \simeq g^{-2/3} \int_0^\eta d\zeta \{ e^{-\zeta} + A_g [1 - e^{-\sqrt{\zeta}} - \sqrt{\zeta}(1 - \sqrt{\zeta}/2) e^{-\sqrt{\zeta}}] \}^{1/2} , \tag{28}$$

where $A_g = 2\sqrt{\pi} g$ for $g = O(1)$ and $A_g = (4\pi g^2)^{2/3}$ for $g \to \infty$. This gives the estimate for the large scale eddy viscosity $\nu_T = \nu(l \to \infty)$:

$$\nu_T \simeq \frac{U_0 d}{32} \times \begin{cases} (2\pi^{1/2} g)^{1/2} , & g = O(1) \\ (4\pi g^2)^{1/3} , & g \to \infty . \end{cases} \tag{29}$$

For small η, $\nu(\eta) \simeq (U_0 d/32)\sqrt{\exp(-\eta) + 4\sqrt{\pi} g \eta}$, implying that the background fluctuations overwhelm the wake velocity defect U_s (leading essentially to the breakup of the wake) at $\eta \sim (4\sqrt{\pi} g)^{-1}$, i.e., at $x \simeq L\sqrt{2\pi}/(\alpha\alpha_\xi)^2$, which is a distance of the order of the mean separation among cylinders. At that distance, the wake thickness is still below the mean cylinder separation L: $l(x) = l_w \sim 0.2L$. The spectral peak of the turbulent fluctuations is reached instead at a distance $x \simeq l_\xi$, corresponding to a wake width $l \sim l_{\text{eddy}} \sim g^{1/4} L \sim (L/d)^{1/4} L$ for $g = O(1)$ and $l_{\text{eddy}} \sim g^{1/3} L \sim (L/d)^{1/3} L$ for $g \to \infty$. At that scale the eddy viscosity reaches its large scale limit; notice that for L/d not too large, l_{eddy} is of the order of the mean cylinder separation.

The interesting result of Eq. (29) is, however, that, contrary to one's intuition, the eddy viscosity ν_T is inversely proportional to the volume fraction of the cylinders: $(d/L)^2$. This is a consequence of the fact that, contrary to three dimensional wakes [6], two dimensional ones remain turbulent at all distances if they are turbulent at the beginning. The basic mechanism for the divergence of ν_T is that when the mean distance among cylinders increases, the drag length l_ξ grows as well, so that at any point, one feels the turbulence from an increasing number of cylinders upstream. Notice that for this reason the limit for $d/L \to 0$ of this problem does not coincide with the isolated cylinder case, and actually is badly behaved, with the eddy viscosity going to infinity. Notice, however, that from Eq. (21), this corresponds to a fluctuation level $U_{l_{\text{eddy}}}/U_0 \propto (d/L)^{2/3}$, which vanishes in the limit, so that the perturbation analysis is still valid.

In most situations of practical interest the cylinders are of finite length and the fluid is bounded by some surface. An interesting example could be that of a flooded forest, in which there is a bottom at some depth h. This generates contributions to the drag ξ and the turbulent velocity. Using the logarithmic profile law [6] the correction to the drag coefficient is given by

$$\delta\xi_B \simeq \frac{\kappa U_0}{h \ln(h/r_0)} , \tag{30}$$

where $\kappa \simeq 0.4$ is the Von Karman constant and r_0 is the length scale of the roughness on the bottom. The typical velocity of the turbulent fluctuations generated by the friction against the bottom is, in the same approximations, $v_B \simeq 0.4 U_0/h[\ln(h/r_0)]$, which, by itself, would lead to a contribution to the eddy viscosity:

$$\nu_B \sim h v_B \sim U_0/\ln(h/r_0) . \tag{31}$$

Using Eqs. (30) and (26) in (29) leads to the obvious result that for $L > h$ the eddy viscosity stops growing with L but tends to the value given by Eq. (31). In this limit, at sufficient distances, the bottom becomes the dominant factor determining the wake profile. In particular, for the range $l(x) < h$ it becomes necessary to use information about the energy spectrum of bottom turbulence [10].

IV. CALCULATION OF TRANSPORT COEFFICIENTS

The expressions for the wake profiles that have been derived in Sec. III can be used, together with the results of Sec. II, to estimate the diffusion coefficients for a passive scalar in terms of the important quantities of the problem, i.e., d, h, and L. Equation (9) in particular is simplified by using Eq. (18) to express U_s in terms of $l(x)$. In these approximations, Eq. (9) reads

$$\langle \Delta x(t)^2 \rangle_w \simeq \sqrt{8\pi}\alpha_U^2 (U_0 d)^2 \bar{n} \int_0^t dt' \int_0^{t'} dt'' \exp[U_0(t'-t'')/l_\xi] \int_{U_0(t'-t'')}^{\infty} dx\, l^{-1}(x) \exp(-2x/l_\xi) \ . \tag{32}$$

Equation (32) gives the longitudinal diffusion due to the presence of the wakes. From the analysis performed in the former section, it appears that the wake shape can be approximated, in rescaled variables, by the expressions $F(\eta)\simeq 2^{1/2}\pi^{1/4}g^{-1/6}\eta$ and $F(\eta)\simeq(4\pi)^{1/3}\eta$ valid, respectively, in the two regimes $g=O(1)$ and $g\to\infty$. Substituting into (32) and using Eqs. (25) and (26) gives the result

$$\langle \Delta x(t)^2 \rangle_w \simeq 4B_g \left[\frac{d}{\alpha^2 \alpha_\xi L} \right]^{1/4} U_0 L \int_0^t dt' \hat{D}(2U_0 t'/l_\xi) \ , \tag{33}$$

where $B_g=1$ for $g=O(1)$ and $B_g =(2\pi^{1/2}g)^{-1/12}\simeq 0.8(\alpha^2\alpha_\xi L/d)^{-1/12}$ for $g\to\infty$. The term $\bar{D}$ is a normalized diffusion coefficient, which tends to a constant value at times of the order of l_ξ/U_0:

$$\hat{D}(x)=\int_0^x d\eta\, e^{\eta/2}\operatorname{erfc}(\sqrt{\eta}) \tag{34}$$

[$\operatorname{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty e^{-t^2}dt$ is the complementary error function]. Using the result $\hat{D}(x\to\infty)\simeq 0.84$ and Eq. (33) gives for the diffusion coefficient $D_w =\lim_{t\to\infty}\langle \Delta x^2(t)\rangle /t$:

$$D_w \simeq C \left[\frac{d}{\alpha^2\alpha_\xi L} \right]^z U_0 L \ , \tag{35}$$

where $C=3.5$, $z=\frac{1}{4}$ for $g=O(1)$ and $C=4.2$, $z=\frac{1}{3}$ for $g\to\infty$. The turbulent diffusivity can be approximated with the expression calculated in the preceding section for the eddy viscosity [see Eq. (29)]:

$$D_T \sim C'(\alpha^2\alpha_\xi L/d)z' U_0 d \ , \tag{36}$$

where $C'=0.01$, $z'=\frac{1}{2}$ for $g=O(1)$ and $C'=0.007$, $z'=\frac{2}{3}$ for $g\to\infty$. This should be compared with the result inside the wake of an isolated cylinder: $D_T\sim U_0 d/32\simeq 0.03U_0 d$. Notice that both longitudinal and transversal diffusivity depend on the same combination of adjustable parameters $\hat{\alpha}=\alpha^2\alpha_\xi$.

The analysis leading to the expression for the eddy viscosity given by Eq. (29) showed that the inhomogeneity of the turbulent field did not play a role at large scales. The same result can be obtained in a more intuitive way, modeling turbulent diffusion as diffusion from quenched disorder [11]. If turbulence were homogeneous and isotropic, diffusion could be estimated by an eddy diffusivity: $l_{\rm eddy}v$, where $l_{\rm eddy}$ is the scale of the energy containing eddies. The effect of nonhomogeneity is to bring about a distribution of values of $l_{\rm eddy}$. Dividing the space into boxes having the size of the energy containing eddy at that point, a tracer particle will be scattered to a neighboring box in an eddy turnover time. This leads to the estimate for the mean diffusivity:

$$D\sim\frac{\Delta x^2}{\Delta t}\sim\left[\frac{\sum_i N_i l_i^2 t_i}{\sum_i N_i t_i}\right]\left[\frac{\sum_i N_i t_i^2}{\sum_i N_i t_i}\right]^{-1}=\frac{\sum_i N_i l_i^2 t_i}{\sum_i N_i t_i^2} \ , \tag{37}$$

where N_i is the number of eddies of type i, which have size l_i and turnover time t_i, while $P_i=N_i/\sum_i N_i t_i$ is the actual probability to find a tracer in an eddy of type i. However, the place where the nonhomogeneity occurs is the near wake region, where the eddies are smaller and faster and their contribution is killed in the weighed sum of Eq. (37).

Analysis of Eqs. (35) and (36) shows clearly that longitudinal diffusion, due mainly to the wakes, dominates over the transverse part, due to turbulence. This result is strengthened by the fact that the very near wake has not been considered in the analysis.

The results from the Appendix allow one to estimate also the skewness of the concentration profiles; in the same approximation of Eq. (32), Eq. (A3) leads to the expression

$$\langle \Delta x(t)^3 \rangle_w \simeq 18\bar{n}\sqrt{2\pi}\alpha_U^3 U_0 d^3 \int_0^t dt' \int_0^\infty dx \exp(-x/l_\xi) \left[\int_{x/U_0}^{x/U_0+t-t'} dt''\, l^{-1}(U_0 t'') \exp(-U_0 t''/l_\xi) \right]^2 . \tag{38}$$

Analysis of Eqs. (32) and (38) shows that diffusion approaches normality, on a time scale of the order of l_ξ/U_0, with the skewness ratio $S=\langle \Delta x^3\rangle/\langle \Delta x^2\rangle^{3/2}$ decaying like $t^{-1/2}$.

The analysis carried on so far disregarded the effect of trapping by the close wake regions. It is important to estimate the limit of this approximation, concerning especially the prediction of normal diffusion on time scales of the order of the transit time over a length l_ξ. If the distribution of residence times in the trapping regions has a finite mean, a simple set of equations, including the effect of trapping, can be introduced:

$$\begin{aligned}(\partial\rho_1/\partial t)&=(\rho-\rho_1)/\tau_{\rm tr}\\ (\partial/\partial t+U_0\partial/\partial x-\nabla\cdot\mathbf{D}\cdot\nabla)\rho&=c_{\rm tr}(\rho_1-\rho)/\tau_{\rm tr} \ .\end{aligned} \tag{39}$$

In these equations, $\mathbf{D}$ is the diffusion tensor with longitudinal and transversal components given by Eqs. (35) and (36) (and cross terms equal to zero), ρ_1 is the concentration of contaminant in the trapping regions, $c_{\rm tr}$ is the volume fraction occupied by these regions, and τ_{tr} is the trapping time. Equation (39) can be solved for a point source at time zero in the right hand side of the second of (39), in terms of moments of ρ and ρ_1 with respect to x. It is then a matter of lengthy but straightforward algebra to show that for times $t > \tau_{tr}$ and $c_{tr} \ll 1$

$$\langle \Delta x^2 \rangle \simeq (D_w + 2c_{tr} U_0^2 \tau_{tr}) t \ , \qquad \langle \Delta x^3 \rangle \simeq 6 c_{tr} U_0 \tau_{tr} (D_w + 5 U_0^2 \tau_{tr}) t \ , \tag{40}$$

which indicates that the condition for the trapping effect to be negligible in the calculation of the skewness is $\tau_{tr} < l_\xi / U_0$. Notice that the trapping effect produces a correction also in the diffusion coefficient; comparing with Eq. (35), we see that the trapping effect can be neglected only if $c_{tr} U_0 \tau_{tr} < (d/L)^{1/4} L$. [Incidentally, the same reasoning used before to illustrate how, in our case, turbulence inhomogeneity does not affect the eddy diffusivity, shows that the trapping regions could become dominant in Eq. (37) due to the divergence of the corresponding scattering time $t_i \sim \tau_{tr}$.] Since $c_{tr} \propto (d/L)^2$, while τ_{tr} is independent of d/L, it appears that the condition for the trapping effect to be negligible, both in the analysis of diffusion and of skewness, is that the cylinder separation be large. To understand the exact range of applicability of this approximation requires, however, knowledge of the parameters τ_{tr} and $(L/d)^2 c_{tr}$, which is not provided by the present model.

V. DISCUSSION

The analysis carried on in this work has permitted me to obtain some predictions on the scaling of the longitudinal and transverse diffusion coefficients with respect to the volume ratio of the cylinders: $(d/L)^2$. The most important result is the divergence of the diffusion coefficients in the $d/L \to 0$ limit. The actual magnitude of these coefficients still depends on a single free parameter $\hat{\alpha}$, which contains the deviations from the isolated cylinder case, in the drag, in the eddy viscosity dependence on the turbulent velocity, and in the ratio between turbulent velocity and mean velocity differences. These results provide an interesting example of the long distance character of the hydrodynamic forces in two dimensional configurations. In fact, also in the case of laminar wakes, a similar situation occurs. To see this, it is sufficient to remember that, because of dimensional requirements, laminar wakes in two dimensions can depend on the cylinder radii only through logarithms [12]. Hence, making in Eq. (31) the substitution $d \to \nu_0 / U_0$, with ν_0 the Newtonian viscosity, and taking $l(x) \sim \sqrt{\nu_0 x / U_0}$, leads to the approximate scaling for the wake diffusion: $D_\omega \propto U_0 L$.

It must be stressed that this model is based on some assumptions and approximations, which are not completely under control. In particular, it is not clear whether a condition in the form $U_l \sim v_l$ [see Eq. (20) and paragraph preceding it] is acceptable, when there are turbulent fluctuations at scales larger than l. A similar problem occurs when analyzing the dynamics of $U_1(\mathbf{r}|\mathbf{r}')$ at distances such that U_1 is smaller than the turbulent fluctuations; the difficulty arises at the moment of separating out an eddy viscosity containing the effect of eddies at scales below $l(|\mathbf{r}-\mathbf{r}'|)$, and also because the effect on U_1 of the larger scale energetic eddies (which are going to be present unless $|\mathbf{r}-\mathbf{r}'| > l_\xi$) is not taken into consideration. The basic hope is that the errors generated by these approximations be bounded in some way, so that, at least in the limit $d/L \to 0$, it could be possible to reabsorb them in a redefinition of the free constants.

It would be interesting at this point to investigate the practical relevance of the asymptotic regime considered in this paper to the situation in typical environments, in which the ratio d/L and the cylinder length are both finite. The first thing that occurs away from the limit $d/L \to 0$, as is apparent from Eq. (40), is that the contribution to diffusion coming from the effect of trapping ceases to be negligible with respect to that coming from the inhomogeneity of the large scale velocity field. In particular, the contribution to the non-Gaussian character of the diffusion process produced by these regions becomes dominant. This suggests a crossover between two completely different mechanisms of diffusion at some finite (and probably small) value of d/L. As far as the finite length of the cylinders is concerned, the basic effect is the presence of boundary layers coming from either the ends of the cylinders or from any surfaces bounding the flow. In this case, there could be a regime in which the longitudinal diffusion is still produced by the wakes, while the transverse part is due to turbulence from the boundary layers. These issues are both subjects of work in progress.

ACKNOWLEDGMENTS

I would like to thank F. Villalba and M. Garcia of the Comitè Ecuadoriano Energìa Atòmica for interesting and helpful conversation and L. Kadanoff of the University of Chicago, for his encouragement in carrying on this research. This research was supported in part by the University of Chicago Material Research Laboratory.

APPENDIX

The same line of arguments used to derive the diffusion coefficient given in Eq. (9) can be used to calculate the skewness:

$$\langle \Delta x(t)^3 \rangle = \langle x_1(t)^3 \rangle + 2\langle x_1(t) \rangle^3 - 3\langle x_1(t) \rangle \langle x_1(t)^2 \rangle \ .$$

This quantity depends on the three-point correlation for the density, $\langle nnn \rangle$, which can be expressed in the following form:

$$\langle n(\mathbf{r}_1) n(\mathbf{r}_2) n(\mathbf{r}_3) \rangle = \bar{n}^3 P(\mathbf{r}_1, \mathbf{r}_2 | \mathbf{r}_3) + \bar{n}^2 [P(\mathbf{r}_1|\mathbf{r}_2)\delta(\mathbf{r}_1 - \mathbf{r}_3) + P(\mathbf{r}_2|\mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r}_2) + P(\mathbf{r}_1|\mathbf{r}_3)\delta(\mathbf{r}_3 - \mathbf{r}_2)] + 3\bar{n}\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_1 - \mathbf{r}_3) \ , \tag{A1}$$

where $P(\mathbf{r}_1,\mathbf{r}_2|\mathbf{r}_3)$ is the conditional probability of finding a cylinder in $\mathbf{r}$ given the presence of a cylinder in $\mathbf{r}_1$ and another one in $\mathbf{r}_2$. [This quantity satisfies the symmetry relation: $P(\mathbf{r}_1,\mathbf{r}_2|\mathbf{r}_3)=P(\mathbf{r}_2,\mathbf{r}_3|\mathbf{r}_1)=P(\mathbf{r}_1,\mathbf{r}_3|\mathbf{r}_2)$ and has the limit for $|\mathbf{r}_1-\mathbf{r}_2|\to\infty$: $P(\mathbf{r}_1,\mathbf{r}_2|\mathbf{r}_3)=P(\mathbf{r}_1|\mathbf{r}_3)P(\mathbf{r}_2|\mathbf{r}_3)$.] Substituting Eq. (8) into the expression for the wake dependent part of the skewness gives the result

$$\langle \Delta x(t)^3\rangle_w=\int_0^t dt_1 dt_2 dt_3 \int d^2r_1 d^2r_2 d^2r_3[\bar{n}^3 w_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)+3\bar{n}^2 w_2(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_2-\mathbf{r}_3)+3\bar{n}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_2-\mathbf{r}_3)] \times U_1(\mathbf{r}_1|\mathbf{r}(t_1))U_1(\mathbf{r}_2|\mathbf{r}(t_2))U_1(\mathbf{r}_3|\mathbf{r}(t_3)) , \tag{A2}$$

where $w_2(\mathbf{r}_1-r_2)=P(\mathbf{r}_1|\mathbf{r}_2)-1$ and $w_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=P(\mathbf{r}_1,\mathbf{r}_2|\mathbf{r}_3)+2-[P(\mathbf{r}_1|\mathbf{r}_2)+P(\mathbf{r}_2|\mathbf{r}_3)+P(\mathbf{r}_3|\mathbf{r}_1)]$. As in the case of w_2, if the distribution of cylinders is sufficiently random, w_3 is nonzero only for separation distances well below L, so that only terms linear in $\bar{n}$ survive in (A2), leading to the expression

$$\langle \Delta x(t)^3\rangle_w \simeq 18\bar{n}\int_0^{x(t)} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_{-\infty}^{x_3} dx \int_{-\infty}^{+\infty} dy U_0^{-3} U_1(\mathbf{r}|\mathbf{r}_1)U_1(\mathbf{r}|\mathbf{r}_2)U_1(\mathbf{r}|\mathbf{r}_3) . \tag{A3}$$

Again, the contribution from the very near wake has been disregarded and this would have produced a divergent contribution. At the same time Eq. (A3) is dominated by the far wake region provided $l(x)\propto x^\alpha$ with $\alpha<1$. The analysis carried on in Sec. IV shows, however, that the very near wake contribution to transport, disregarded in (A3), could become dominant away from the dilute limit.

[1] J. C. R. Hunt, Annu. Rev. Fluid Mech. **17**, 447 (1985).
[2] H. C. Brinkman, Appl. Sci. Res. **A1**, 27 (1947).
[3] E. J. Hinch, J. Fluid Mech. **83**, 695 (1977).
[4] D. L. Koch and J. F. Brady, J. Fluid Mech. **154**, 399 (1984).
[5] Y. Kaneda, J. Fluid Mech. **167**, 455 (1986).
[6] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1972).
[7] It should be mentioned that the convergence to a mean field regime was guaranteed, in the case of Stokes flows, by linearity. In the turbulent case it is not completely clear how the correlations decay away in the dilute limit, and one has to take the applicability of a mean approach as one of the assumptions of the model.
[8] IAEA Report No. SM-319/9, 1991 (unpublished).
[9] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[10] A. E. Perry, S. Henbest, and M. S. Chong, J. Fluid Mech. **165**, 163 (1986).
[11] J. F. Bouchaud and A. Georges, Phys. Rep. **195**, 128 (1990).
[12] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1989).